Initial-State Coulomb Interaction in the $dd \to \alpha\pi^0$ Reaction

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The effects of initial-state Coulomb interactions in the charge-symmetry-breaking reaction $dd \to \alpha\pi^0$ are investigated within a previously published formalism. This is a leading order effect in which the Coulomb interaction between the two initial state protons leads to the breakup of the two deuterons into a continuum state that is well connected to the final $\alpha\pi^0$ state by the strong emission of a pion. As a first step, we use a simplified set of d and α wave functions and a plane-wave approximation for the initial dd state. This Coulomb mechanism, by itself, yields cross sections that are much larger than the experimental ones, and which are comparable in size to the contributions from other mechanisms. Inclusion of this mechanism is therefore necessary in a realistic calculation.

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I. INTRODUCTION

The concepts of charge independence and charge symmetry provide powerful tools in organizing the multiplet structure of systems of hadrons and nuclei. These symmetries are not perfect; diverse small but interesting violations have been discovered; see the reviews of Refs. [1, 2]. Our concern here is with charge symmetry breaking.

Hadronic states can be regarded as approximately charge symmetric, i.e. invariant under a rotation by 180° around the 2-axis in isospin space. Charge symmetry (CS) is a subset of the general isospin symmetry, charge independence (CI), which requires invariance under *any* rotation in isospin space. In Quantum Chromodynamics (QCD), CS requires that the dynamics are unchanged under the exchange of the up and down quarks [1]. In the language of hadrons, this symmetry translates into e.g. the invariance of the strong interaction under the exchange of protons and neutrons. However, since the up and down quarks do have different masses ($m_u \neq m_d$) [3, 4], the QCD Lagrangian is not charge symmetric and neither are the strong interactions of hadrons. This symmetry violation is called charge symmetry breaking (CSB). The different electromagnetic interactions of the up and down quarks also contribute to CSB.

Observing the effects of CSB interactions therefore provides a probe of m_u and m_d, which are fundamental, but poorly known, parameters of the standard model. For example, the light quark mass difference causes the neutron to be heavier than the proton. If this were not the case, our universe would be very different, as a consequence of the dependence of Big-Bang nucleosynthesis on the relative abundances of protons and neutrons. Experimental evidence for CSB has been demonstrated many times, see e.g. Refs. [1, 2]. Two exciting recent observations of CSB in experiments involving the production of neutral pions have stimulated current interest: Many years of effort led to the observation of CSB in $np \to d\pi^0$ at TRIUMF. After a careful treatment of systematic errors, the CSB forward-backward asymmetry of the differential cross section was found to be $A_{\rm fb} = (17.2 \pm 8 \pm 5.5) \times 10^{-4}$ [5], where the former error is statistical and the latter systematical. In addition, the final experiment at the IUCF Cooler ring has reported a very convincing $dd \to \alpha\pi^0$ signal near threshold ($\sigma = 12.7 \pm 2.2$ pb at $T_d = 228.5$ MeV and 15.1 ± 3.1 pb at 231.8 MeV) [6]. These data are consistent with the pion being produced in an s-wave, as expected from the proximity of the threshold ($T_d = 225.6$ MeV). Studies of the $dd \to \alpha\pi^0$ reaction thus present exciting new opportunities for developing the understanding of CSB.

The reaction $dd \to \alpha\pi^0$ obviously violates isospin conservation, but more specifically, it violates charge symmetry since the deuterons and the α-particle are self-conjugate under the charge-symmetry operator, with a positive eigenvalue, while the neutral pion wave function changes sign. This reaction could thus not occur if charge symmetry were conserved, and its cross section is proportional to the square of the CSB amplitude. This phenomenon is unique, because all other observations of CSB involve interferences with charge-symmetric amplitudes.

Due to the recent availability of high-quality experimental data on CSB, a theoretical interpretation using fundamental CSB mechanisms is called for. At momenta comparable to the pion mass, $Q \sim m_\pi$, QCD and its symmetries (and in particular CSB) can be described by a hadronic effective field theory (EFT), called Chiral Perturbation Theory (χPT), for extensive reviews see Refs. [7, 8, 9]. This EFT has been extended to pion production in Refs. [10, 11, 12, 13, 14] where typical momenta are $Q \sim \sqrt{m_\pi M}$, with M as the nucleon mass (see also Ref. [15] where pion production was studied neglecting this large momentum in power counting). The EFT formalism provides specific CSB effects in addition

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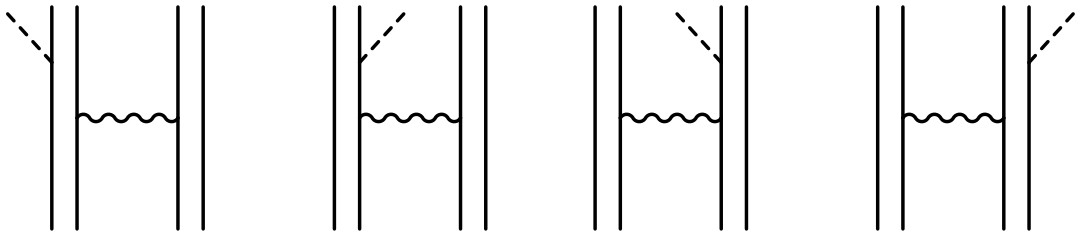

FIG. 1: Diagrams relevant for the inclusion of the Coulomb initial-state interaction. The wiggly lines represent the Coulomb interaction between the two protons of the initial state. The dashed lines represent the emitted pion. For each Coulomb interaction, anyone of the four nucleons may emit the pion.

to the nucleon mass difference. In particular, there are two pion-nucleon seagull interactions related by chiral symmetry to the quark-mass and electromagnetic contributions to the nucleon mass difference [16, 17].

In previous work [18, 19], the cross section for the reaction $dd \to \alpha\pi^0$ was computed near threshold by chiral EFT techniques, using a chiral power counting scheme to assess the expected importance of different interaction terms. The first paper [18] used a plane wave approximation along with Gaussian bound-state wave functions. These initial calculations yielded computed cross sections that are a factor of ~ 2 larger than the measured ones. The effects of initial-state interactions and realistic bound-state wave functions were included later [19], with resulting cross sections of the order of several hundred pb or more. It is thus clear that more work is needed to understand the order of magnitude of the measured cross section, such as a treatment of the effects of loop diagrams along the lines of Ref. [20].

We consider here one specific contribution that has previously been neglected – the influence of the Coulomb interaction in the initial state. The formalism employed is similar to that of Ref. [18]. The main objective is to assess the relevance of this mechanism, so we use simple bound-state wave functions and neglect the effects of strong interactions in the initial state. It is worthwhile to explain some basic features of the calculation that result from invariance principles: Spin, isospin, and symmetry requirements restrict the partial waves allowed for the $dd \to \alpha\pi^0$ reaction. In the spectroscopic notation $^{2S+1}L_J l$, where S, L and J are, respectively, the spin, orbital, and total angular momenta of the dd state, and l denotes the pion angular momentum, the lowest allowed partial waves are $^3P_0 s$ and $^5D_1 p$. Hence, production of an s-wave pion requires that the initial deuterons be in a relative P-wave, with spins coupled to a spin-1 state, forming together a state with zero total angular momentum. On the other hand, a p-wave pion is produced only when the deuterons are in a relative D-wave, with spins maximally aligned to spin 2, requiring either a coupling with $\Delta L = \Delta S = 2$ or D-states of d or α. Interferences between s- and p-waves disappear for any unpolarized observable. We shall therefore be concerned with the production of an s-wave pion. In the mechanism proposed here, the Coulomb interaction between the two protons of different deuterons converts the initial relative P-wave state into an S-wave state. Parity conservation then requires that one of the resulting pairs of nucleons be in a p-wave with orbital angular momentum unity. The strong pseudovector pion production operator then converts this p-wave state into an s-wave state.

This paper is organized in the following manner: The relevant Coulomb mechanism is described in Sect. II, which also explains how the formalism of Ref. [18] is to be employed. We use the simple Gaussian bound-state wave functions of Ref. [18], but extend the calculation by also considering Hulthén wave functions for the deuteron. The detailed evaluation and numerical results are given in Sect. III. For comparison, the effects of Coulomb interactions in the final state are considered in Sec. IV. These have been calculated in Ref. [19] and were found to be very small. Finally, Sect. V assesses our results and discusses how these effects can be included in a realistic calculation that incorporates the strong interactions in the initial state.

II. COULOMB MECHANISM

In the present study, CSB arises from the initial-state Coulomb interaction between the two deuterons, followed by strong pion emission, as shown in Fig. 1. We note that the electromagnetic contributions can be ordered [18] relative to each other in the same fashion as the effects of strong CSB. In this case, the leading order (LO) term considered here is of $\mathcal{O}\left[\alpha_{\rm em} M/(4\pi f_\pi^3 p)\right]$, with M as the nucleon mass. This term is roughly of the same size as the LO strong CSB term which is of $\mathcal{O}\left[\frac{m_d - m_u}{m_d + m_u}\, m_\pi^2/(M 4\pi f_\pi^3 p)\right]$.

The CSB pion production operator $\mathcal{O}_C$ is given by

$$\mathcal{O}_C \;=\; \mathcal{O}_1 \left(E - H_0 + i\epsilon\right)^{-1} V_C, \qquad (1)$$

where V_C is the Coulomb interaction between the two protons in the initial state, which acts to form a four-body continuum state that propagates according to

$(E - H_0 + i\epsilon)^{-1}$. It is convenient to write V_C as a sum of pair-wise operators:

$$V_C = \sum_{j<k=1,4} Q_j Q_k v_C^{j,k}, \quad (2)$$

where the $Q_{j,k}$ are nucleon charge operators. The operator H_0 is the sum of the kinetic energies of each of the four nucleons. The strong pion production operator is denoted by $\mathcal{O}_1$ and is given by

$$\begin{aligned} \mathcal{O}_1 &= \frac{g_A}{2f_\pi} \sum_i \tau_{z,i}\, \boldsymbol{\sigma}_i \cdot \Big[\boldsymbol{q}_i - \frac{\omega}{2M} (\boldsymbol{k}_i^{'} + \boldsymbol{k}_i) \Big] \\ &\rightarrow \left(-\frac{g_A}{2f_\pi} \right) \frac{\mu}{M} \sum_i \tau_{z,i}\, \boldsymbol{\sigma}_i \cdot \boldsymbol{k}_i, \end{aligned} \quad (3)$$

where the $\boldsymbol{k}_i, \boldsymbol{k}_i^{'}$ are nucleon momenta before and after the pion emission, respectively. The p-wave term with $\boldsymbol{q}_i = -\boldsymbol{p}_\pi$ can be ignored in the threshold regime considered here. The factor ω is also replaced by the pion mass $\mu = 134.974$ MeV.

The present analysis uses a plane-wave approximation and the simplest possible d and α bound-state wave functions, those of a Gaussian form. Assuming spatially symmetric bound-state wave functions, the invariant amplitude is given by

$$\mathcal{M} = \int d^3r d^3\rho_1 d^3\rho_2 \, \langle A|\mathcal{O}|DD\rangle, \quad (4)$$

with

$$|A\rangle = \sqrt{2E_\alpha}\, \Psi_\alpha(\boldsymbol{r}, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\, |\alpha\rangle, \quad (5)$$

$$|DD\rangle = \sqrt{s}\, \Phi_d(\boldsymbol{\rho}_1)\Phi_d(\boldsymbol{\rho}_2)\, |dd\rangle, \quad (6)$$

where Ψ_α and Φ_d are the spatial parts of the α-particle and deuteron bound-state wave functions, and $s = 4E_d^2$ is the total c.m. energy squared. The ket vectors $|\alpha\rangle$ and $|dd\rangle$ contain the fully anti-symmetrized spin and isospin wave functions. Because of symmetry requirements, the plane-wave dd scattering wave function is included in $|dd\rangle$ as given by Eqs. (16) and (17) below. The invariant amplitude can then be written as

$$\begin{aligned} \mathcal{M} &= \sqrt{2E_\alpha s} \int d^3r d^3\rho_1 d^3\rho_2 \, \Psi_\alpha^\dagger(\boldsymbol{r}, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \\ &\quad \langle\alpha|\mathcal{O}|dd\rangle \, \Phi_d(\boldsymbol{\rho}_1)\Phi_d(\boldsymbol{\rho}_2), \end{aligned} \quad (7)$$

The matrix element $\langle\alpha|\mathcal{O}|dd\rangle$ contains all the spin-isospin couplings of the nucleons and the pion production operator $\mathcal{O}$. The wave functions are expressed in terms of the (2+2) Jacobian coordinates

$$\begin{aligned} \boldsymbol{R} &= \frac{1}{4}(\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 + \boldsymbol{r}_4) \quad (\equiv 0 \text{ in c.m.}), \\ \boldsymbol{r} &= \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2 - \boldsymbol{r}_3 - \boldsymbol{r}_4), \\ \boldsymbol{\rho}_1 &= \boldsymbol{r}_1 - \boldsymbol{r}_2, \\ \boldsymbol{\rho}_2 &= \boldsymbol{r}_3 - \boldsymbol{r}_4, \end{aligned} \quad (8)$$

with the corresponding momenta

$$\begin{aligned} \boldsymbol{K} &= \boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4 \quad (\equiv 0 \text{ in c.m.}), \\ \boldsymbol{k} &= \frac{1}{2}(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4) \\ &= \frac{1}{2}(\boldsymbol{p}_1 - \boldsymbol{p}_2) \quad (\equiv \boldsymbol{p} \text{ in c.m.}), \\ \boldsymbol{\kappa}_1 &= \frac{1}{2}(\boldsymbol{k}_1 - \boldsymbol{k}_2), \\ \boldsymbol{\kappa}_2 &= \frac{1}{2}(\boldsymbol{k}_3 - \boldsymbol{k}_4), \end{aligned} \quad (9)$$

defined so that $\sum_i \boldsymbol{k}_i \cdot \boldsymbol{r}_i = \boldsymbol{K}\cdot\boldsymbol{R} + \boldsymbol{k}\cdot\boldsymbol{r} + \boldsymbol{\kappa}_1\cdot\boldsymbol{\rho}_1 + \boldsymbol{\kappa}_2\cdot\boldsymbol{\rho}_2$. The Jacobians are equal to unity in both representations.

The ground-state wave functions are represented by Gaussian functions, and these may be explicitly expressed in the above coordinates using $\sum_{i<j}(\boldsymbol{r}_i - \boldsymbol{r}_j)^2 = 4\boldsymbol{r}^2 + 2\boldsymbol{\rho}_1^2 + 2\boldsymbol{\rho}_2^2$, yielding

$$\Psi_\alpha(\boldsymbol{r}, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \frac{8}{\pi^{9/4}\alpha^{9/2}} \exp\left[-\frac{\left(2\boldsymbol{r}^2 + \boldsymbol{\rho}_1^2 + \boldsymbol{\rho}_2^2\right)}{\alpha^2} \right], \quad (10)$$

$$\Phi_d(\boldsymbol{\rho}_i) = \frac{1}{\pi^{3/4}\beta^{3/2}} \exp\left(-\frac{\boldsymbol{\rho}_i^2}{2\beta^2} \right), \; i = 1, 2 \quad (11)$$

where the parameters $\alpha = 2.77$ fm and $\beta = 3.189$ fm have been fixed using the measured α and d rms point radii $\langle r_\alpha^2\rangle^{1/2} = 1.47$ fm and $\langle r_d^2\rangle^{1/2} = 1.953$ fm [21]. We shall work in momentum space and therefore record the corresponding wave functions

$$\widetilde{\Psi}_\alpha(\boldsymbol{k}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) = N_\alpha \exp\left[-\frac{\alpha^2}{8}(\boldsymbol{k}^2 + 2\boldsymbol{\kappa}_1^2 + 2\boldsymbol{\kappa}_2^2) \right],$$

$$N_\alpha \equiv \frac{\alpha^{9/2}}{8\pi^{9/4}}, \quad (12)$$

$$\widetilde{\Phi}_d(\kappa_i) = N_d \exp\left(-\frac{\boldsymbol{\kappa}_i^2\beta^2}{2} \right), \quad i = 1, 2$$

$$N_d \equiv \left(\frac{\beta^2}{\pi} \right)^{3/4}. \quad (13)$$

In order to study the sensitivity of our results to the choice of wave functions, we also use a deuteron wave function of the Hulthén form:

$$\widetilde{\Phi}_d^H(\boldsymbol{\kappa}_i) = N_d^H \left(\frac{1}{\boldsymbol{\kappa}_i^2 + a^2} - \frac{1}{\boldsymbol{\kappa}_i^2 + b^2} \right), \quad i = 1, 2$$

$$N_d^H = \frac{\sqrt{ab(a+b)}}{\pi(a-b)}, \quad (14)$$

where the parameters are given by $a = 0.23161$ fm^{-1} and $b = 1.3802$ fm^{-1} [22].

Since we have assumed that the orbital parts of the wave functions are symmetric under the exchange of any pair of nucleons, we may define the initial- and final-state spin-isospin wave functions as

$$|\alpha\rangle = \frac{1}{\sqrt{2}} \{((1,2)_1, (3,4)_1)_0 \, [[1,2]_0, [3,4]_0]_0$$

$$- ((1,2)_0,(3,4)_0)_0\,[[1,2]_1,[3,4]_1]_0\}\,,$$

$$\equiv \frac{1}{\sqrt{2}}\left(|\alpha_1\rangle + |\alpha_2\rangle\right) \quad (15)$$

$$|dd\rangle = \frac{1}{\sqrt{3}}\left(1 - P_{23} - P_{24}\right)|d_{12}d_{34}\rangle\,, \quad (16)$$

$$|d_{12}d_{34}\rangle = ((1,2)_1,(3,4)_1)_S\,[[1,2]_0,[3,4]_0]_0 \times\frac{1}{\sqrt{2}}\left[e^{i\boldsymbol{p}\cdot\boldsymbol{r}} + (-)^L e^{-i\boldsymbol{p}\cdot\boldsymbol{r}}\right], \quad (17)$$

where $(i,j)_s$ and $[i,j]_T$ are the spin and isospin Clebsch-Gordan couplings, with magnetic quantum numbers suppressed, for nucleons, or nucleon pairs, i and j coupling to spin s and isospin T, respectively. We shall refer to the first term of Eq. (15) as the "*dd*" component of the α because the pairs (12) and (34) each have the spin and isospin of the deuteron. In the above equations, P_{ij} is the permutation operator of the indicated nucleons. The symmetry requirements for the exchange of the deuterons are represented by the (orbital angular momentum dependent) combination of plane waves in Eq. (17), with $\boldsymbol{p}$ as the relative momentum of the deuterons. Even though the expression for the α state superficially singles out a (12)+(34) configuration, it is indeed fully antisymmetric in all indices. This particular form is used because it closely matches the form of the initial-state wave functions, thereby simplifying the evaluation of the spin-isospin summations in the matrix element. In practice, the *dd* wave function can be simplified to

$$|dd\rangle = \sqrt{6}\,((1,2)_1,(3,4)_1)_S\,[[1,2]_0,[3,4]_0]_0\; e^{i\boldsymbol{p}\cdot\boldsymbol{r}}, \quad (18)$$

since each of the three terms in Eq. (16) gives an identical contribution to the matrix element.

The expressions (15) through (18) provide insight that simplifies the calculation: The Coulomb interaction has no spin operator, so the initial state is connected only with the "*dd*" component of the alpha particle, which means that only the Class III [1] part of the Coulomb operator $(\tau_{z,i}+\tau_{z,j})$ contributes. One of these τ_z operators finds another from within the pion production operator and is squared to $\pm$ unity. As a result, the matrix element turns out to be proportional to the spin operators of the (1,2) and (3,4) systems. In the normalization used here, the spin-averaged cross section (for s-wave pions) is given by

$$\sigma = \frac{1}{16\pi s}\frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{p}|}\,\frac{1}{9}\sum_{\rm pol.}|\mathcal{M}|^2, \quad (19)$$

where the summation is over the deuteron polarizations.

III. EVALUATION

The analysis is most conveniently performed in momentum space. Combination of Eq. (1) with Eq. (4) yields, upon Fourier transformation, the Coulomb contribution of present interest, $\mathcal{M}_C$:

$$\mathcal{M}_C = \sqrt{E_\alpha s}\left(-\frac{g_A}{2f_\pi}\right)\frac{\mu}{M}\int d^3k\,d^3\kappa_1 d^3\kappa_2 \times \langle\alpha_1|\widetilde{\Psi}_\alpha(\boldsymbol{k},\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)\sum_{i=1,4}\frac{\tau_{z,i}\,\boldsymbol{\sigma}_i\cdot\boldsymbol{k}_i}{E - \frac{2\boldsymbol{\kappa}_1^2+2\boldsymbol{\kappa}_2^2+\boldsymbol{k}^2}{2M} + i\epsilon} \times (\boldsymbol{k},\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2|V_C|dd,\boldsymbol{p}\rangle, \quad (20)$$

where the relation $\sum_{j=1,4}\boldsymbol{k}_j^2 = 2\boldsymbol{\kappa}_1^2 + 2\boldsymbol{\kappa}_2^2 + \boldsymbol{k}^2$ has been applied in the free propagator, and $|dd,\boldsymbol{p}\rangle$ represents the initial *dd* (relative plane wave) state of Eq. (18), including the internal spatial, spin, and isospin degrees of freedom. Further, $(\boldsymbol{k},\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2|V_C|dd,\boldsymbol{p}\rangle$ denotes the momentum space representation of the state formed by the action of V_C on the initial state. The round bracket notation used here signifies that only the spatial degrees of freedom are included.

A. Reduction to Quadrature

The first step in the calculation is the simplification of the pion production operator in Eq. (20). Define the operator X according to

$$X = \sum_{i,j<k=1,4}\tau_{z,i}\,\boldsymbol{\sigma}_i\cdot\boldsymbol{k}_i\,Q_jQ_k\,v_C^{j,k} \quad (21)$$

$$= (\tau_{z,3}\,\boldsymbol{\sigma}_3\cdot\boldsymbol{k}_3 + \tau_{z,4}\,\boldsymbol{\sigma}_4\cdot\boldsymbol{k}_4)\,\frac{(1+\tau_{z,3})}{2}\left[\frac{(1+\tau_{z,1})}{2}\,v_C^{3,1} + \frac{(1+\tau_{z,2})}{2}\,v_C^{3,2}\right]$$
$$+ (\tau_{z,3}\,\boldsymbol{\sigma}_3\cdot\boldsymbol{k}_3 + \tau_{z,4}\,\boldsymbol{\sigma}_4\cdot\boldsymbol{k}_4)\,\frac{(1+\tau_{z,4})}{2}\left[\frac{(1+\tau_{z,1})}{2}\,v_C^{4,1} + \frac{(1+\tau_{z,2})}{2}\,v_C^{4,2}\right]$$
$$+ (\tau_{z,1}\,\boldsymbol{\sigma}_1\cdot\boldsymbol{k}_1 + \tau_{z,2}\,\boldsymbol{\sigma}_2\cdot\boldsymbol{k}_2)\,\frac{(1+\tau_{z,1})}{2}\left[\frac{(1+\tau_{z,3})}{2}\,v_C^{3,1} + \frac{(1+\tau_{z,4})}{2}\,v_C^{4,1}\right]$$
$$+ (\tau_{z,1}\,\boldsymbol{\sigma}_1\cdot\boldsymbol{k}_1 + \tau_{z,2}\,\boldsymbol{\sigma}_2\cdot\boldsymbol{k}_2)\,\frac{(1+\tau_{z,2})}{2}\left[\frac{(1+\tau_{z,3})}{2}\,v_C^{3,2} + \frac{(1+\tau_{z,4})}{2}\,v_C^{4,2}\right]. \quad (22)$$

It is instructive to consider the first term of Eq. (22): It should be noted that the operators $v_C^{3,1}, v_C^{3,2}$ do not flip the spin of their deuteron. Also, the initial-state deuteron $(1,2)$ is connected to the deuteron-like $(1,2)$ component of the α. Thus the terms with $\tau_{z,1}, \tau_{z,2}$ can be dropped, and the initial-state deuteron $(3,4)$ is similarly connected to the deuteron-like $(3,4)$ component of the α. We are required to have CSB, so only the terms proportional to $\tau_{z,3}$ are relevant. For the $(3,4)$ "deuteron" of the final state we have $\tau_{z,3} = -\tau_{z,4}$. Thus the first term of Eq. (22) simplifies to $\frac{1}{4}(\boldsymbol{\sigma}_3 \cdot \boldsymbol{k}_3 - \boldsymbol{\sigma}_4 \cdot \boldsymbol{k}_4)(v_C^{3,1} + v_C^{3,2})$. Similar manipulation of the remaining terms in Eq. (22) leads to the result

$$X \;=\; \left(\frac{\boldsymbol{\sigma}_3 \cdot \boldsymbol{k}_3 - \boldsymbol{\sigma}_4 \cdot \boldsymbol{k}_4}{4}\right)\left[v_C^{3,1} + v_C^{3,2} - v_C^{4,1} - v_C^{4,2}\right] \;+\; \left(\frac{\boldsymbol{\sigma}_1 \cdot \boldsymbol{k}_1 - \boldsymbol{\sigma}_2 \cdot \boldsymbol{k}_2}{4}\right)\left[v_C^{3,1} + v_C^{4,1} - v_C^{3,2} - v_C^{4,2}\right]. \quad (23)$$

Next define the spin operators $\boldsymbol{S}_1 = \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2), \boldsymbol{S}_2 = \frac{1}{2}(\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_4), \boldsymbol{\Sigma}_1 = \frac{1}{2}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2), \boldsymbol{\Sigma}_2 = \frac{1}{2}(\boldsymbol{\sigma}_3 - \boldsymbol{\sigma}_4)$, such that each of the $\boldsymbol{\sigma}_i$ is a linear combination of the $\boldsymbol{S}_i$ and $\boldsymbol{\Sigma}_i$. Only the terms proportional to $\boldsymbol{S}_i$ connect the initial state to the "dd" component of the α. Thus one finds

$$X \;\to\; \frac{\boldsymbol{S}_2 \cdot \boldsymbol{\kappa}_2}{2}\left[v_C^{3,1} + v_C^{3,2} - v_C^{4,1} - v_C^{4,2}\right] \;+\; \frac{\boldsymbol{S}_1 \cdot \boldsymbol{\kappa}_1}{2}\left[v_C^{3,1} + v_C^{4,1} - v_C^{3,2} - v_C^{4,2}\right]. \quad (24)$$

Further, it is permissible to interchange indices 3 and 4 in the spatial wave functions multiplying the first term of Eq. (24), and similarly to interchange 1 and 2 in those multiplying the second term. The final form of the operator X is thus

$$X \;\to\; \boldsymbol{S}_2 \cdot \boldsymbol{\kappa}_2\left[v_C^{3,1} + v_C^{3,2}\right] \;+\; \boldsymbol{S}_1 \cdot \boldsymbol{\kappa}_1\left[v_C^{3,1} + v_C^{4,1}\right]. \quad (25)$$

The next task is to compute the momentum space matrix element of the operator X. This is given by

$$(\boldsymbol{k}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2|\, X\, |dd, \boldsymbol{p}) \;=\; \boldsymbol{S}_2 \cdot \boldsymbol{\kappa}_2\; (\boldsymbol{k}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2|\, (v_C^{3,1} + v_C^{3,2})\, |dd, \boldsymbol{p}) \;+\; \boldsymbol{S}_1 \cdot \boldsymbol{\kappa}_1\; (\boldsymbol{k}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2|\, (v_C^{3,1} + v_C^{4,1})\, |dd, \boldsymbol{p}), \quad (26)$$

with the spatial matrix elements

$$(\boldsymbol{k}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2|\, v_C^{j,k}\, |dd, \boldsymbol{p}) \;=\; \int \frac{d^3\rho_1 d^3\rho_2 d^3r}{(2\pi)^{9/2}}\; e^{-i\boldsymbol{\kappa}_1\cdot\boldsymbol{\rho}_1 - i\boldsymbol{\kappa}_2\cdot\boldsymbol{\rho}_2 - i\boldsymbol{k}\cdot\boldsymbol{r}}\; \frac{\alpha_{\rm em}}{|\boldsymbol{r}_j - \boldsymbol{r}_k|}\; e^{i\boldsymbol{p}\cdot\boldsymbol{r}}\; \Phi_d(\boldsymbol{\rho}_1)\Phi_d(\boldsymbol{\rho}_2), \quad (27)$$

which are thus found to be products of the momentum-space Coulomb interaction with deuteron wave functions evaluated at shifted values of the momentum. In particular, we define $\boldsymbol{v} \equiv \boldsymbol{k} - \boldsymbol{p}$ and obtain

$$\begin{aligned}
(\boldsymbol{k}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2|\, v_C^{3,1}\, |dd, \boldsymbol{p}) \;&=\; \frac{4\pi\alpha_{\rm em}}{(2\pi)^{3/2}\boldsymbol{v}^2}\; \widetilde{\Phi}_d\left(\boldsymbol{\kappa}_1 - \frac{\boldsymbol{v}}{2}\right)\widetilde{\Phi}_d\left(\boldsymbol{\kappa}_2 + \frac{\boldsymbol{v}}{2}\right), \\
(\boldsymbol{k}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2|\, v_C^{3,2}\, |dd, \boldsymbol{p}) \;&=\; \frac{4\pi\alpha_{\rm em}}{(2\pi)^{3/2}\boldsymbol{v}^2}\; \widetilde{\Phi}_d\left(\boldsymbol{\kappa}_1 + \frac{\boldsymbol{v}}{2}\right)\widetilde{\Phi}_d\left(\boldsymbol{\kappa}_2 + \frac{\boldsymbol{v}}{2}\right), \\
(\boldsymbol{k}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2|\, v_C^{4,1}\, |dd, \boldsymbol{p}) \;&=\; \frac{4\pi\alpha_{\rm em}}{(2\pi)^{3/2}\boldsymbol{v}^2}\; \widetilde{\Phi}_d\left(\boldsymbol{\kappa}_1 - \frac{\boldsymbol{v}}{2}\right)\widetilde{\Phi}_d\left(\boldsymbol{\kappa}_2 - \frac{\boldsymbol{v}}{2}\right).
\end{aligned} \quad (28)$$

Insertion of these results into Eq. (26) finally gives

$$(\boldsymbol{k}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2|\, X\, |dd, \boldsymbol{p}) \;=\; \frac{8\pi\alpha_{\rm em}}{(2\pi)^{3/2}\boldsymbol{v}^2}\; (\boldsymbol{S}_2 \cdot \boldsymbol{\kappa}_2 - \boldsymbol{S}_1 \cdot \boldsymbol{\kappa}_1)\; \widetilde{\Phi}_d\left(\boldsymbol{\kappa}_1 + \frac{\boldsymbol{v}}{2}\right)\widetilde{\Phi}_d\left(\boldsymbol{\kappa}_2 + \frac{\boldsymbol{v}}{2}\right). \quad (29)$$

Examination of Eq. (20) reveals that $\boldsymbol{p}$ is the only momentum remaining after the integrals have been performed. Thus the terms with $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ in Eq. (29) both end up being proportional to $\widehat{\boldsymbol{p}}$, which we may take as the $z-$axis. Furthermore, as the integrands of both terms are identical, the whole operator must be proportional to $(S_{2z} - S_{1z})$ and the integrand to $\frac{\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2}{2}$. We thus need to consider the spin matrix element

$$\begin{aligned}
\langle 1M_1, 1M_2|0,0\rangle\langle 1M_1, 1M_2|(S_{2z} - S_{1z})|1M_1, 1M_2\rangle \;&=\; \langle 1M_1, 1M_2|0,0\rangle(M_2 - M_1) \\
&=\; 2M_2\langle 1M_1, 1M_2|0,0\rangle,
\end{aligned} \quad (30)$$

where $\langle 1M_1, 1M_2|0,0\rangle$ is the Clebsch-Gordan coefficient that couples the spins in the "dd" component to zero. Armed with this knowledge, we may now use Eqs. (18) and (29,30) in the matrix element (20) to obtain

$$\begin{aligned}
\mathcal{M}_C \;&=\; \sqrt{6E_\alpha s}\left(-\frac{\mu g_A}{f_\pi}\right)\frac{8\pi\alpha_{\rm em}}{(2\pi)^{3/2}} \;\times\; M_2\langle 1M_1, 1M_2|0,0\rangle \quad (31) \\
&\times\; \int d^3k d^3\kappa_1 d^3\kappa_2\; \frac{(\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2)\cdot\widehat{\boldsymbol{p}}}{\boldsymbol{v}^2}\; \left[2ME - (2\boldsymbol{\kappa}_1^2 + 2\boldsymbol{\kappa}_2^2 + \boldsymbol{k}^2) + i\epsilon\right]^{-1}\; \widetilde{\Psi}_\alpha(\boldsymbol{k}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)\, \widetilde{\Phi}_d\left(\boldsymbol{\kappa}_1 + \frac{\boldsymbol{v}}{2}\right)\; \widetilde{\Phi}_d\left(\boldsymbol{\kappa}_2 + \frac{\boldsymbol{v}}{2}\right).
\end{aligned}$$

The above equation is our main result, and it allows for the use of general radial wave functions. However, because of the zeros in the energy denominator, it may not be well suited to evaluation using Monte-Carlo techniques. Nevertheless, if certain simple wave functions are used, Eq. (31) may be simplified further. In the next subsection, this will be performed for wave functions of the Gaussian and Hulthén types.

B. Gaussian and Hulthén Deuteron Wave Functions

If the Gaussian wave functions of Eqs. (12) and (13) are used, the expression (31) becomes

$$\mathcal{M}_C = \sqrt{6E_\alpha s}\left(-\frac{\mu g_A}{f_\pi}\right)\frac{8\pi\alpha_{\rm em}}{(2\pi)^{3/2}} \times M_2\langle 1M_1,1M_2|0,0\rangle \times N_a N_d^2 \times I_g, \tag{32}$$

$$I_g = \int d^3k d^3\kappa_1 d^3\kappa_2 \; \frac{(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2)\cdot\widehat{\boldsymbol{p}}}{\boldsymbol{v}^2} \; \left[2ME-(2\boldsymbol{\kappa}_1^2+2\boldsymbol{\kappa}_2^2+\boldsymbol{k}^2)+i\epsilon\right]^{-1} \times \exp\left[-\frac{\alpha^2}{8}(\boldsymbol{k}^2+2\boldsymbol{\kappa}_1^2+2\boldsymbol{\kappa}_2^2)\right] \exp\left[-\frac{\beta^2}{2}\left(\boldsymbol{\kappa}_1+\frac{\boldsymbol{v}}{2}\right)^2\right] \exp\left[-\frac{\beta^2}{2}\left(\boldsymbol{\kappa}_2+\frac{\boldsymbol{v}}{2}\right)^2\right], \tag{33}$$

where we recall the definition $\boldsymbol{v}\equiv\boldsymbol{k}-\boldsymbol{p}$. The factors in the denominator of Eq. (33) may be rewritten in terms of Gaussians, giving

$$\boldsymbol{v}^{-2} = \int_0^\infty d\gamma \, \exp\left[-\gamma(\boldsymbol{v}^2+\epsilon_1)\right], \tag{34}$$

$$\left[2ME-(2\boldsymbol{\kappa}_1^2+2\boldsymbol{\kappa}_2^2+\boldsymbol{k}^2)+i\epsilon\right]^{-1} = -i\int_0^\infty d\nu \, \exp\left[\, i\nu\left(2ME-(2\boldsymbol{\kappa}_1^2+2\boldsymbol{\kappa}_2^2+\boldsymbol{k}^2)+i\epsilon\right)\right], \tag{35}$$

where the regulator ϵ assures that the integral over ν converges, and ϵ_1 is included to handle the point $\boldsymbol{k}=\boldsymbol{p}$. The use of the above identities leads to an 11 dimensional integral, of which 9 dimensions involve products of Gaussian functions, such that these integrals may be computed analytically by successive completion of squares in the exponents. This procedure yields a two-dimensional integral over ν and γ, which is then computed numerically. In this way, using the definitions

$$\boldsymbol{\kappa}\equiv\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2, \quad \boldsymbol{l}\equiv\frac{\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2}{2}, \quad \boldsymbol{\kappa}_1=\frac{\boldsymbol{\kappa}}{2}+\boldsymbol{l}, \quad \boldsymbol{\kappa}_2=\frac{\boldsymbol{\kappa}}{2}-\boldsymbol{l}, \quad \boldsymbol{\kappa}_1^2+\boldsymbol{\kappa}_2^2=2\boldsymbol{l}^2+\frac{\boldsymbol{\kappa}^2}{2}, \tag{36}$$

the integral I_g may be re-written as

$$I_g = -i\int_0^\infty d\nu\, d\gamma \int d^3v\, d^3\kappa\, d^3l \; \boldsymbol{\kappa}\cdot\widehat{\boldsymbol{p}} \; \exp\left[-\frac{(\boldsymbol{v}+\boldsymbol{p})^2\alpha^2}{8}-\left(2\boldsymbol{l}^2+\frac{\boldsymbol{\kappa}^2}{2}\right)\left(\frac{\alpha^2}{4}+\frac{\beta^2}{2}\right)\right] \times \exp\left[-\frac{\boldsymbol{\kappa}\cdot\boldsymbol{v}\beta^2}{2}-\frac{\boldsymbol{v}^2\beta^2}{4}-\gamma(\boldsymbol{v}^2+\epsilon_1)\right] \exp\left[i\nu\left(2ME-4\boldsymbol{l}^2-\boldsymbol{\kappa}^2-(\boldsymbol{v}+\boldsymbol{p})^2+i\epsilon\right)\right], \tag{37}$$

where the completion of squares is facilitated by the definitions

$$\bar{\alpha}^2\equiv\alpha^2+8i\nu, \quad R_l^2\equiv\frac{\alpha^2}{2}+\beta^2+4i\nu, \quad R_v^2\equiv\frac{R_l^2}{4}+\gamma-\frac{\beta^4}{4R_l^2}, \quad E=\frac{\boldsymbol{p}^2}{M_d}=\frac{\boldsymbol{p}^2}{2M}. \tag{38}$$

For all successive equations, we will define $I_j=-i|\boldsymbol{p}|\pi^{9/2}I_{1j}$. At this point, the regulators ϵ and ϵ_1 may be safely dropped. By combination of the above results and definitions, we find for the case of Gaussian deuteron wave functions

$$\mathcal{M}_C = \sqrt{6E_\alpha s}\left(-\frac{\mu g_A}{f_\pi}\right)\frac{8\pi\alpha_{\rm em}}{(2\pi)^{3/2}} \times M_2\langle 1M_1,1M_2|0,0\rangle \times N_a N_d^2 \times (-i|\boldsymbol{p}|\pi^{9/2}) \times I_{1g},$$

$$I_{1g} = \int_0^\infty d\nu\, d\gamma \, \frac{\beta^2\bar{\alpha}^2}{R_l^8 R_v^5} \exp\left\{-\boldsymbol{p}^2\left[\frac{\alpha^2}{8}\left(1-\frac{\alpha^2}{8R_v^2}\right)+\frac{\nu^2}{R_v^2}\right]\right\} \exp\left(\frac{i\nu\boldsymbol{p}^2\alpha^2}{4R_v^2}\right). \tag{39}$$

If the Hulthén wave function, given in Eq. (14), is used for the deuteron, the product of deuteron wave functions in Eq. (31) may be re-written as

$$\widetilde{\Phi}_d^h\left(\boldsymbol{\kappa}_1+\frac{\boldsymbol{v}}{2}\right)\widetilde{\Phi}_d^h\left(\boldsymbol{\kappa}_2+\frac{\boldsymbol{v}}{2}\right) = \left(N_d^h\right)^2\int_0^\infty d\eta_1 d\eta_2 \left\{\exp\left[-\eta_1\left(\boldsymbol{\kappa}_1+\frac{\boldsymbol{v}}{2}\right)^2\right]\left(e^{-\eta_1 a^2}-e^{-\eta_1 b^2}\right)\right\} \times \left\{\exp\left[-\eta_2\left(\boldsymbol{\kappa}_2+\frac{\boldsymbol{v}}{2}\right)^2\right]\left(e^{-\eta_2 a^2}-e^{-\eta_2 b^2}\right)\right\}, \tag{40}$$

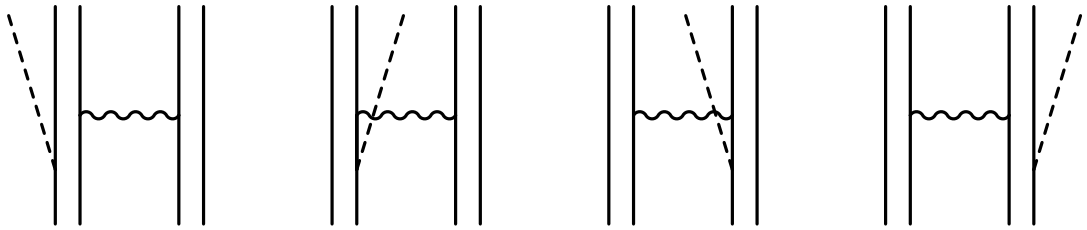

FIG. 2: Diagrams relevant for the inclusion of the Coulomb final-state interaction. The wiggly lines represent the Coulomb interaction between the two protons in the final state. The dashed lines represent the emitted pion.

after which the calculation proceeds, as before, through successive completion of squares, but in the Hulthén case we are left with a four-dimensional integral suitable for numerical evaluation. The end result is that the integral I_{1g} of Eq. (39) should be replaced by I_{1h}, which is given by

$$I_{1h} \;=\; \int_0^\infty d\nu\, d\gamma\, d\eta_1 d\eta_2\, \frac{\eta_{12}\bar{\alpha}^2}{8R_l^3 R_\kappa^5 R_v^5}\, \exp\left\{-\boldsymbol{p}^2\left[\frac{\alpha^2}{8}\left(1-\frac{\alpha^2}{8R_v^2}\right)+\frac{\nu^2}{R_v^2}\right]\right\}\, \exp\left(\frac{i\nu\boldsymbol{p}^2\alpha^2}{4R_v^2}\right)\, f_{ab}(\eta_1,\eta_2), \tag{41}$$

where the definitions

$$\eta_{12} = \frac{\eta_1+\eta_2}{4} - \frac{(\eta_1-\eta_2)^2}{4R_l^2}, \quad R_l^2 = \frac{\bar{\alpha}^2}{2}+\eta_1+\eta_2, \quad R_\kappa^2 = \frac{\bar{\alpha}^2}{8}+\eta_{12}, \quad R_v^2 = R_\kappa^2+\gamma-\frac{\eta_{12}^2}{R_\kappa^2}, \tag{42}$$

$$f_{ab}(\eta_1,\eta_2) \;=\; e^{-a^2(\eta_1+\eta_2)} + e^{-b^2(\eta_1+\eta_2)} - 2\, e^{-a^2\eta_1} e^{-b^2\eta_2}, \tag{43}$$

are used. These expressions represent the complete amplitude. In order to obtain the cross section, it is necessary to evaluate the spin sum

$$\sum_{M_1,M_2} M_2^2\, \langle 1M_1, 1M_2|0,0\rangle^2 \;=\; \frac{2}{3} \tag{44}$$

and insert everything into Eq. (19), yielding

$$\sigma_g \;=\; E_\alpha \left(\frac{\mu g_A}{f_\pi}\right)^2 \frac{|\boldsymbol{p}||\boldsymbol{p}_\pi|\alpha_{\rm em}^2}{288}\, \frac{\alpha^9\beta^6}{\sqrt{\pi}}\, \left|I_{1g}\right|^2, \tag{45}$$

$$\sigma_h \;=\; E_\alpha \left(\frac{\mu g_A}{f_\pi}\right)^2 \frac{|\boldsymbol{p}||\boldsymbol{p}_\pi|\alpha_{\rm em}^2}{288}\, \frac{\alpha^9}{\pi^{3/2}}\, \frac{a^2b^2(a+b)^2}{(a-b)^4}\, \left|I_{1h}\right|^2, \tag{46}$$

where σ_g and σ_h again denote the expressions relevant for the Gaussian and Hulthén deuteron wave functions, respectively. It should also be noted that the above expressions remain valid for the Coulomb interaction in the ^{4}He bound state, which is considered in Sect. IV.

IV. COULOMB IN THE ^{4}HE BOUND STATE

A complete assessment of all Coulomb effects should include a treatment of the Coulomb interactions in both the initial and final states. Our focus here is on the initial-state effects, as these have not been considered up to now. However, it is also worthwhile to compute the effects of the Coulomb interactions in the final state within the present framework. We recall that in this framework, the strong interaction between the initial-state deuterons is neglected, and simple bound-state wave functions are used. If the effects of Coulomb interactions in the ^{4}He bound state are included, the reaction $dd \to \alpha\pi^0$ can proceed via strong pion production, which is here assumed to be initiated by the one-body operator. The relevant CBS pion production operator is then given by

$$\mathcal{O}_C^F \;=\; V_C\, (-\epsilon_B - H_0 + i\epsilon)^{-1}\, \mathcal{O}_1, \tag{47}$$

where $\epsilon_B \simeq 28.3$ MeV is the ^{4}He binding energy. It is instructive to define the operator Y according to

$$
\begin{aligned}
Y &= \sum_{i,j<k=1,4} Q_j Q_k \, v_C^{j,k} \, \tau_{z,i} \, \boldsymbol{\sigma}_i \cdot \boldsymbol{k}_i \qquad (48)\\
&= \frac{(1+\tau_{z,3})}{2}\left[\frac{(1+\tau_{z,1})}{2}\, v_C^{3,1} + \frac{(1+\tau_{z,2})}{2}\, v_C^{3,2}\right](\tau_{z,3}\boldsymbol{\sigma}_3\cdot\boldsymbol{k}_3 + \tau_{z,4}\boldsymbol{\sigma}_4\cdot\boldsymbol{k}_4)\\
&+ \frac{(1+\tau_{z,4})}{2}\left[\frac{(1+\tau_{z,1})}{2}\, v_C^{4,1} + \frac{(1+\tau_{z,2})}{2}\, v_C^{4,2}\right](\tau_{z,3}\boldsymbol{\sigma}_3\cdot\boldsymbol{k}_3 + \tau_{z,4}\boldsymbol{\sigma}_4\cdot\boldsymbol{k}_4)\\
&+ \frac{(1+\tau_{z,1})}{2}\left[\frac{(1+\tau_{z,3})}{2}\, v_C^{3,1} + \frac{(1+\tau_{z,4})}{2}\, v_C^{4,1}\right](\tau_{z,1}\boldsymbol{\sigma}_1\cdot\boldsymbol{k}_1 + \tau_{z,2}\boldsymbol{\sigma}_2\cdot\boldsymbol{k}_2)\\
&+ \frac{(1+\tau_{z,2})}{2}\left[\frac{(1+\tau_{z,3})}{2}\, v_C^{3,2} + \frac{(1+\tau_{z,4})}{2}\, v_C^{4,2}\right](\tau_{z,1}\boldsymbol{\sigma}_1\cdot\boldsymbol{k}_1 + \tau_{z,2}\boldsymbol{\sigma}_2\cdot\boldsymbol{k}_2). \qquad (49)
\end{aligned}
$$

In the above equation, the one-body operator can either maintain the spin-parity-isospin quantum numbers of a single deuteron, or produce a single two-nucleon state with $S=0$. As the operators $v_C^{3,1}, v_C^{3,2}$ do not flip the spin of their two-nucleon system, the quantum numbers of the dd state must be maintained. Manipulations similar to those of Sect. III lead to the simplification

$$
Y = \left[v_C^{3,1} + v_C^{3,2} - v_C^{4,1} - v_C^{4,2}\right]\left(\frac{\boldsymbol{\sigma}_3\cdot\boldsymbol{k}_3 - \boldsymbol{\sigma}_4\cdot\boldsymbol{k}_4}{4}\right) + \left[v_C^{3,1} + v_C^{4,1} - v_C^{3,2} - v_C^{4,2}\right]\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{k}_1 - \boldsymbol{\sigma}_2\cdot\boldsymbol{k}_2}{4}\right), \qquad (50)
$$

and finally to

$$
Y \;\rightarrow\; \left[v_C^{3,1} + v_C^{3,2}\right]\boldsymbol{S}_2\cdot\boldsymbol{\kappa}_2 + \left[v_C^{3,1} + v_C^{4,1}\right]\boldsymbol{S}_1\cdot\boldsymbol{\kappa}_1, \qquad (51)
$$

which is analogous to that of Eq. (25) for the initial-state Coulomb interaction. Computation of the momentum space matrix element then leads to the result

$$
(\boldsymbol{k},\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2|Y|dd,\boldsymbol{p}) = \frac{8\pi\alpha_{\rm em}}{(2\pi)^{3/2}\boldsymbol{v}^2}\left[\boldsymbol{S}_2\cdot\left(\boldsymbol{\kappa}_2+\frac{\boldsymbol{v}}{2}\right) - \boldsymbol{S}_1\cdot\left(\boldsymbol{\kappa}_1+\frac{\boldsymbol{v}}{2}\right)\right]\widetilde{\Phi}_d\left(\boldsymbol{\kappa}_1+\frac{\boldsymbol{v}}{2}\right)\widetilde{\Phi}_d\left(\boldsymbol{\kappa}_2+\frac{\boldsymbol{v}}{2}\right). \qquad (52)
$$

Since $\boldsymbol{p}$ is again the only momentum remaining after integration, the above matrix element may be treated along the same lines as Eq. (29). Thus it is again possible to extract a factor $(S_{2z} - S_{1z})$, giving finally

$$
\begin{aligned}
\mathcal{M}_C^F &= \sqrt{6E_\alpha s}\left(-\frac{\mu g_A}{f_\pi}\right)\frac{8\pi\alpha_{\rm em}}{(2\pi)^{3/2}} \times M_2\langle 1M_1,1M_2|0,0\rangle \times N_a N_d^2 \times I_g^F\\
I_g^F &= \int d^3k d^3\kappa_1 d^3\kappa_2 \; \frac{(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{v})\cdot\widehat{\boldsymbol{p}}}{\boldsymbol{v}^2} \; \left[-2M\epsilon_B - (2\boldsymbol{\kappa}_1^2 + 2\boldsymbol{\kappa}_2^2 + \boldsymbol{k}^2)\right]^{-1}\\
&\times \exp\left[-\frac{\alpha^2}{8}(\boldsymbol{k}^2 + 2\boldsymbol{\kappa}_1^2 + 2\boldsymbol{\kappa}_2^2)\right]\exp\left[-\frac{\beta^2}{2}\left(\boldsymbol{\kappa}_1+\frac{\boldsymbol{v}}{2}\right)^2\right]\exp\left[-\frac{\beta^2}{2}\left(\boldsymbol{\kappa}_2+\frac{\boldsymbol{v}}{2}\right)^2\right], \qquad (53)
\end{aligned}
$$

where the Gaussian wave functions of Eqs. (12) and (13) have been employed. We proceed by writing the last two factors of Eq. (33) in terms of Gaussians, and note that the only difference with the general procedure of the previous section is that we may use

$$
\left[-2M\epsilon_B - (2\boldsymbol{\kappa}_1^2 + 2\boldsymbol{\kappa}_2^2 + \boldsymbol{k}^2)\right]^{-1} = -\int_0^\infty d\nu \, \exp\left[-\nu\left(2M\epsilon_B + 2\boldsymbol{\kappa}_1^2 + 2\boldsymbol{\kappa}_2^2 + \boldsymbol{k}^2\right)\right], \qquad (54)
$$

upon which I_g^F becomes an 11-dimensional integral, of which 9 can again be computed analytically by completion of squares in the exponents. Analogously to the previous section, we employ the notation

$$
\tilde{\alpha}^2 \equiv \alpha^2 + 8\nu, \quad \tilde{R}_l^2 \equiv \frac{\alpha^2}{2} + \beta^2 + 4\nu, \quad \tilde{R}_v^2 \equiv \frac{\tilde{R}_l^2}{4} + \gamma - \frac{\beta^4}{4\tilde{R}_l^2}, \qquad (55)
$$

along with the definition $I_j^F = |\boldsymbol{p}|\pi^{9/2} I_{1j}^F$. The matrix element in Eq. (53) then becomes

$$
\begin{aligned}
\mathcal{M}_C^F &= \sqrt{6E_\alpha s}\left(-\frac{\mu g_A}{f_\pi}\right)\frac{8\pi\alpha_{\rm em}}{(2\pi)^{3/2}} \times M_2\langle 1M_1,1M_2|0,0\rangle \times N_a N_d^2 \times |\boldsymbol{p}|\pi^{9/2} \times I_{1g}^F,\\
I_{1g}^F &= \int_0^\infty d\nu \, d\gamma \, \frac{\tilde{\alpha}^4}{2\tilde{R}_l^8\tilde{R}_v^5} \exp\left[-\frac{\boldsymbol{p}^2\tilde{\alpha}^2}{8}\left(1 - \frac{\tilde{\alpha}^2}{8\tilde{R}_v^2}\right)\right]\exp\left(-2\nu\, M\epsilon_B\right). \qquad (56)
\end{aligned}
$$

The evaluation of the final-state Coulomb mechanism of the preceding subsection can also be implemented using the Hulthén wave functions for the deuteron, given in Eq. (14). The net result is that the integral I_{1g}^F should be replaced by I_{1h}^F, with

$$I_{1h}^F \;=\; \int_0^\infty d\nu\, d\gamma\, d\eta_1 d\eta_2\, \frac{\tilde{\alpha}^4}{64\tilde{R}_l^3\tilde{R}_\kappa^5\tilde{R}_v^5}\, \exp\left[-\frac{\boldsymbol{p}^2\tilde{\alpha}^2}{8}\left(1-\frac{\tilde{\alpha}^2}{8\tilde{R}_v^2}\right)\right]\, \exp\left(-2\nu\, M\epsilon_B\right)\, f_{ab}(\eta_1,\eta_2), \tag{57}$$

where the definitions

$$\zeta_{12}=\frac{\eta_1+\eta_2}{4}-\frac{(\eta_1-\eta_2)^2}{4\tilde{R}_l^2},\quad \tilde{R}_l^2=\frac{\tilde{\alpha}^2}{2}+\eta_1+\eta_2,\quad \tilde{R}_\kappa^2=\frac{\tilde{\alpha}^2}{8}+\zeta_{12},\quad \tilde{R}_v^2=R_\kappa^2+\gamma-\frac{\zeta_{12}^2}{\tilde{R}_\kappa^2}, \tag{58}$$

are used. The cross sections can then be computed using the expressions given in the previous section on the initial-state Coulomb interaction.

V. NUMERICAL RESULTS AND DISCUSSION

The model parameters used in the present calculations are given in Table I, and the calculated cross-sections at $T_d = 228.5$ MeV and $T_d = 231.8$ MeV are summarized in Table II, such that σ_g and σ_h denote the results for the initial-state Coulomb interaction, obtained with Gaussian and Hulthén deuteron wave functions. If Gaussian wave functions are used throughout, the results are 59 pb and 75 pb at the two energies considered. If Hulthén wave functions are used instead for the deuterons, these results increase to 87 pb and 111 pb. In either case, the effects of the initial-state Coulomb interaction are significant, as the experimental values are 12.7 and 15.1 pb, respectively. It should also be noted that these differences are much smaller than those encountered between the CD-Bonn and Argonne V18 potentials in Ref. [19]. The first toy-model calculations yielded nominal values of 23 pb and 30 pb. The present mechanism is therefore clearly large enough to warrant inclusion in a fully realistic calculation.

TABLE I: Summary of parameters used in the calculation. The values of α and β, which appear in the expressions for the Gaussian bound-state wave functions, are from Ref. [21], whereas a and b are relevant for the Hulthén deuteron, and have been taken from Ref. [22].

α [fm]	2.770
β [fm]	3.189
a [fm^{-1}]	0.23161
b [fm^{-1}]	1.3802
$M\epsilon_B$[fm^{-2}]	0.68
f_π [MeV]	92.4
g_A	1.26
$\alpha_{\rm em}^{-1}$	137.04

The results for the Coulomb interaction in the final state are denoted σ_g^F and σ_h^F, and are also given in Table II. If Gaussian wave functions are used throughout, the results are much smaller, about 1 pb, which represents $\sim 1\%$ of those found for the Coulomb interaction in the initial state. The use of Hulthén deuteron wave functions in the initial state is found to enhance the effects of the final-state Coulomb interaction. However, they are still relatively small, about 10% of those of the initial-state Coulomb interaction.

TABLE II: Summary of momenta, calculated cross-sections and integrals for the matrix elements. The quantities with the subscripts g and h correspond to the Gaussian and Hulthén deuteron wave functions, respectively. The energies and momenta correspond to those of Ref. [6].

	$T_d = 228.5$ MeV	$T_d = 231.8$ MeV
p [MeV]	462.913	466.924
p_π [MeV]	19.372	28.266
E_α [MeV]	3727.430	3727.487
I_{1g} [fm^{-5}]	$(1.3638, 1.3919)\times 10^{-5}$	$(1.2440, 1.3213)\times 10^{-5}$
I_{1g}^F [fm^{-5}]	2.3043×10^{-6}	2.1713×10^{-6}
I_{1h} [fm^{-1}]	$(2.7011, 2.1617)\times 10^{-3}$	$(2.4919, 2.0821)\times 10^{-3}$
I_{1h}^F [fm^{-1}]	1.1465×10^{-3}	1.0957×10^{-3}
σ_g [pb]	58.95	75.25
σ_h [pb]	85.77	111.2
σ_g^F [pb]	0.824	1.077
σ_h^F [pb]	9.419	12.66

The principal result of this study is the manifest need to incorporate the effects of Coulomb interactions in the initial state into the realistic calculation that includes strong initial-state interactions. Recent progress in the treatment of Coulomb interactions in few-body scattering calculations [23] should eventually allow such computations to be performed.

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